STEIN FILLABLE 3-MANIFOLDS ADMIT POSITIVE OPEN BOOK DECOMPOSITIONS ALONG ARBITRARY LINKS

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Abstract. It is known by A. Loi and R. Piergallini that a closed, oriented, smooth 3-manifold is Stein fillable if and only if it has a positive open book decomposition. In the present paper we will show that for every link L in a Stein fillable 3-manifold there exists an additional knot L' to L such that the link $L \cup L'$ is the binding of a positive open book decomposition of the Stein fillable 3-manifold. To prove the assertion, we will use the divide, which is a generalization of real morsification theory of complex plane curve singularities, and 2-handle attachings along Legendrian curves.

1. **Introduction**

An open book decomposition of a closed, oriented, smooth 3-manifold is the following: Let S be a compact, oriented, smooth 2-dimensional manifold with boundary ∂S and h an automorphism of S which is the identity on ∂S. If a closed, oriented, smooth 3-manifold M can be obtained from $S \times [0,1]$ by identifying $(h(x),0)$ and $(x,1)$ for $x \in S$ and $(y,0)$ and (y,t) for $y \in \partial S$ and all $t \in [0,1]$, then we say that M has an *open book decomposition*. The manifold S is called a *fiber surface* and ∂S the *binding*. Note that the binding is a *fibered link* in M. An open book decomposition of M is said *positive* if its monodromy h consists of a product of positive Dehn twists.

In [S] J.R. Stallings proved that for every link L in S^3 there exists a knot L' such that $L \cup L'$ is the binding of an open book decomposition of S^3. Moreover he proved that the knot L' can be chosen in such a way that L' and each connected component of L has an arbitrarily prescribed linking number. As a corollary, we can conclude that for every link L in a closed, oriented, smooth 3-manifold M there exists a knot L' such that $L \cup L'$ is the binding of an open book decomposition of M, see [I, Theorem 7.4].

Now we focus on positive open book decompositions of closed, oriented, smooth 3-manifolds. The existence of a positive open book decomposition is related to the following Stein fillability. A complex manifold X is called *Stein* if it admits a strictly plurisubharmonic function $\phi : X \to \mathbb{R}$ which is proper and bounded below. Each level set $\phi^{-1}(c)$, for $c \in \mathbb{R}$, is strictly pseudoconvex, where $\phi^{-1}(c)$ is oriented as the boundary of $\phi^{-1}((-\infty, c])$. In the case where the complex dimension of X is 2, we call $\phi^{-1}((-\infty, c])$

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a *compact Stein surface with boundary*, or we may call it simply a *compact Stein surface*. A closed, oriented, smooth 3-manifold M is called *Stein fillable* if there exists a compact Stein surface $\phi^{-1}((-\infty, c])$ such that $M = \phi^{-1}(c)$. In the paper [E], Y. Eliashberg gave a topological characterization of compact Stein surfaces using handle decompositions along Legendrian curves. Then R.E. Gompf described, in [Gom], the handle decompositions explicitly using framed Legendrian links and Kirby calculus. Following these studies, A. Loi and R. Piergallini represented the manifold $\phi^{-1}((-\infty, c])$ as branched covers of B^4 using framed Legendrian link presentations of handle decompositions. As a consequence, they proved that a closed, oriented, smooth 3-manifold is Stein fillable if and only if it has a positive open book decomposition[Lo-Pi]. Note that a very few examples of non-Stein fillable 3-manifolds are known, see [Lis1, Lis2].

In the present paper we study what kinds of positive open book decompositions a closed, oriented, smooth 3-manifold has. Because of the result of Loi and Piergallini, we can exclude non-Stein fillable 3-manifolds. The main result in this paper is the following:

Theorem 1.1. *Let M be a Stein fillable 3-manifold and fix a link L in M. Then there exists a knot L' in $M \setminus L$ such that the union $L \cup L'$ is the binding of a positive open book decomposition of M.*

In [Gi-I], W. Gibson and the author proved the theorem for the case where M is S^3, and then in [I] the author proved it for the case where M is either $\#nS^1 \times S^2$ or the unit tangent bundles to closed, oriented surfaces. Since a non-Stein fillable 3-manifold does not admit any positive open book decomposition, the result in the present paper is the final solution of this series of studies.

To prove the main theorem we will use *divides* as we did in [Gi-I] and [I]. A divide is the image of a generic, relative immersion of a finite number of copies of the unit interval or the unit circle into the genus-g surface $\Sigma_{g,k}$ with k boundary components. It was originally defined in the unit disk by N. A'Campo in [AC3] as a generalization of real morsification theory of complex plane curve singularities[AC1, AC2, GZ1, GZ2, GZ3]. Each divide in the unit disk determines a link in S^3. In [AC3] A'Campo proved that this link is fibered with positive monodromy diffeomorphism if the divide is connected. To each divide in $\Sigma_{g,k}$ is assigned a link in the unit tangent bundle to $\Sigma_{g,k}$ if $k = 0$, S^3 if $2g + k - 1 = 0$, and the connected sum of $2g + k - 1$ copies of $S^1 \times S^2$ if $2g + k - 1 > 0$. In [I], it is proved that the link is fibered if the divide satisfies a certain condition. In the present paper, we deal with only divides in $\Sigma_{0,n+1}$, whose links are defined in S^3 if $n = 0$ and in $\#nS^1 \times S^2$ if $n > 0$. Hereafter we set $\#0S^1 \times S^2 = S^3$ by definition. The fibration theorem in [I] is proved by constructing a Lefschetz fibration from a compact, connected, oriented 4-manifold to a disk such that the fibration of a divide is induced on its boundary.

For constructing the expected positive open book decomposition of a Stein fillable 3-manifold we first prepare, using a divide in $\Sigma_{0,n+1}$, a Lefschetz fibration over a disk which induces, on its boundary, a positive open book decomposition of $\#nS^1 \times S^2$ whose binding contains the prescribed link. Due to the work of Eliashberg in [E], it is known that a compact Stein surface can be obtained from a 4-ball with 1-handles by attaching 2-handles along Legendrian knots with coefficients the canonical framing minus 1 (see [Gom]). Note that the boundary of the 4-ball with 1-handles is $\#nS^1 \times S^2$. For yielding a Lefschetz fibration after the 2-handle attachings again, we use the method of S. Akbulut and B. Ozbagci

in [Ak-O], that is, we will find a good position of a fiber surface of the positive open book decomposition appearing on the boundary of a Lefschetz fibration such that (i) the links for 2-handle attachings lie on the fiber surface and (ii) their canonical framings coincide with the product framings of the Lefschetz fibration. These two conditions ensure that the 2-handle attachings yield a Lefschetz fibration again and thus, on its boundary, we obtain the Stein fillable 3-manifold equipped with the expected positive open book decomposition.

This paper is organized as follows: In Section 2 we introduce the unit disk D_n in $\mathbb{R}^2$ with n holes and define the 3-manifold $\#nS^1 \times S^2$ as the boundary of a small compact neighborhood of D_n in $\mathbb{C}^2$. In Section 3 we introduce divides in D_n and associated positive open book decompositions of $\#nS^1 \times S^2$. In the end of this section we briefly explain how to see the fiber surfaces of the positive open book decompositions associated with divides. In Section 4 we shortly introduce oriented divides. Section 5 is devoted to the proof of the main theorem. In Section 6 we propose a conjecture for the future.

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2. **Wave fronts on the unit disk with n holes**

Let $(x_1 + iu_1, x_2 + iu_2) \in \mathbb{C}^2$ be the complex valued coordinates of $\mathbb{C}^2$ and set $D := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\} \subset \mathbb{R}^2 \subset \mathbb{C}^2$. We denote by D_n the unit disk with n holes obtained from D by removing n open disks, where $n \geq 0$, and by ∂D_n the boundary of D_n.

We thicken D_n in $\mathbb{C}^2$ as follows: Let A_δ be a small compact neighborhood of ∂D_n in D_n with width $\delta > 0$ and suppose that δ is sufficiently small. Set $B_\delta := D_n \setminus A_\delta$ and thicken it as $\hat{B}_\delta := \{x + iu \in \mathbb{C}^2 \mid x \in B_\delta,\ |u| \leq \delta\}$. Set $\alpha_\delta := \partial A_\delta \setminus \partial D_n$ and thicken A_δ as

$$\hat{A}_\delta := \{x + iu \in \mathbb{C}^2 \mid x \in A_\delta,\ |u|^2 + d(x, \alpha_\delta)^2 \leq \delta^2\},$$

where $d(\cdot, \cdot)$ is the minimal distance of the sets. We then define a thickened disk $N_\delta(D_n)$ of D_n in $\mathbb{C}^2$ by the union $N_\delta(D_n) := \hat{A}_\delta \cup \hat{B}_\delta$, which is a compact 4-dimensional manifold in $\mathbb{C}^2$. We denote the boundary of $N_\delta(D_n)$ by M_n. It is easy to see that M_n is $\#nS^1 \times S^2$, which is the connected sum of n copies of $S^1 \times S^2$ if $n \geq 1$ and S^3 if $n = 0$.

According to the decomposition $N_\delta(D_n) = \hat{A}_\delta \cup \hat{B}_\delta$, we decompose M_n into two pieces, $\partial N_A := \partial \hat{A}_\delta \cap M_n$ and $\partial N_B := \partial \hat{B}_\delta \cap M_n$. The first piece ∂N_A is homeomorphic to a union of solid tori whose core curves are ∂D_n and the second piece ∂N_B is $B_\delta \times S^1$. We may regard the second piece as the unit tangent bundle to B_δ. For the sake of simplicity, we will abbreviate B_δ to B.

When discussing a contact structure in M_n we use the 3-manifold $M'_n := \partial N'_A \cup \partial N_B$ instead of M_n, where $\partial N'_A := \{x + iu \in M_n \mid x \in A_\delta,\ |u|^2 + d(x, \alpha_\delta)^3 = \delta^2\}$, since $|u| = \delta$ and $|u|^2 + d(x, \alpha_\delta)^2 = \delta^2$ are not C^2-differentiable at each point in the boundary of ∂N_A. The complex tangency of $M'_n \subset \mathbb{C}^2$ induces a contact structure ξ in M'_n and we can check by direct calculation that if $\delta > 0$ is sufficiently small then M'_n is strictly pseudoconvex. Thus ξ is the standard contact structure in $\#nS^1 \times S^2$ compatible with its standard

orientation. Here the standard contact structure is the unique tight contact structure in $\#nS^1 \times S^2$ (up to isotopy). Note that the uniqueness is known in [B, E2].

A *contact element* on B is a line tangent to B at a point in B. We consider co-oriented contact elements on B. For each point $x \in B$ and local coordinates (x_1, x_2) in B centered at x, a co-oriented contact element is represented by the equation

$$\alpha_x := (cos\theta)dx_1 + (\sin\theta)dx_2 = 0,$$

where the direction θ is the co-orientation of the contact element. The co-oriented contact element is a point in the unit cotangent bundle to B and it naturally corresponds to a unit tangent vector in the unit tangent bundle to B. Thus the space of co-oriented contact elements on B corresponds to the unit tangent bundle to B, which is the 3-manifold ∂N_B. In ∂N_B, the contact form α of ξ is given by $\alpha|_{\partial N_B} = -(u_1 dx_1 + u_2 dx_2)$, where $u_1^2 + u_2^2 = \delta^2$, and hence the contact structure ξ of $\partial N_B \subset M'_n$ coincides with the contact structure determined by the co-oriented contact elements on B.

A link in a 3-manifold equipped with a contact structure is called *Legendrian* if all the tangent vectors to the link lie on the 2-plane field of the contact structure.

A *wave front* w in B is the image of a (generic) immersion of a finite number of copies of the unit circle into B, possibly with cusps, equipped with co-orientation. Each line tangent to w with co-orientation is a co-oriented contact element and those contact elements constitute a Legendrian link L in ∂N_B. We say that L is the *Legendrian link of* w, or w is a *wave front representative of* L.

Lemma 2.1. *Every Legendrian link in M_n has a wave front representative on B up to Legendrian isotopy.*

Proof. Since the piece ∂N_A is the union of solid tori, using Legendrian isotopy we can assume that the Legendrian link stays in ∂N_B. Then the assertion is a well-known fact in contact topology. □

3. Fibration theorem of divides in D_n

In this section we introduce divides in D_n and their links and fibrations in $\#nS^1 \times S^2$. The idea is based on the works of A'Campo in [AC3, AC4]. Divides in the genus-g surface, possibly with boundary, have been studied in [I], which contain divides in D_n.

Definition 3.1. A *divide* P in D_n is the image of a generic, relative immersion of a finite number of copies of the unit interval or the unit circle into D_n. Each image of the unit interval (resp. circle) is called an *interval (resp. circle) component* of P. The generic and relative conditions are the following:

(i) the image has neither self-tangent points nor triple points;
(ii) each endpoint of an interval component lies on ∂D_n and the interval component intersects ∂D_n at the endpoints transversely;
(iii) a circle component does not intersect ∂D_n.

An *edge* of P is the closure of a connected component of $P \setminus \{$*double points*$\}$ and a *region* of P is a connected component of $D_n \setminus P$. If a region of P is bounded by only P

then it is called an *interior region*, and otherwise it is called an *exterior region*. For each exterior region, its intersection with ∂D_n is called the *outside boundary*.

Definition 3.2. A divide P in D_n is *admissible* if it satisfies the following:

(iv) P is connected;
(v) each interior region of P is simply connected;
(vi) each exterior region of P is either (a) simply connected and the outside boundary in that region is connected*, or (b) an annulus such that one boundary component is a component of $\partial\Sigma_{g,n}$ and the other is contained in P;
(vii) P allows a checkerboard coloring, that is, a coloring with two colors, black and white, such that if two regions of P share an edge of P in their boundaries then they are painted with different colors.

A Morse function $f : \mathbb{R}^2 \to \mathbb{R}$ is a function which has only quadratic singularities. A *maximum* (resp. *saddle* and *minimum*) of f is a quadratic singularity with Morse index 0 (resp. 1 and 2). A *level set* (or an *r-level set*) of f is the set in $\mathbb{R}^2$ given by $X_r := \{x \in \mathbb{R}^2 \mid f(x) = r\}$ for $r \in \mathbb{R}$. In particular, if r is a regular value then X_r consists of disjoint smooth curves in $\mathbb{R}^2$ and if r is a critical value of only saddle singularities then X_r consists of immersed curves in $\mathbb{R}^2$.

Definition 3.3. Let P be an admissible divide in D_n and fix a checkerboard coloring of $D_n \setminus P$. A *Morse function f_P associated with P* is a Morse function $f_P : \mathbb{R}^2 \to \mathbb{R}$ which satisfies the following:

(1) the 0-level set of f_P coincides with P in D_n and, in particular, $0 \in \mathbb{R}$ is either a regular value or a critical value of only saddle singularities of f_P;
(2) each interior region of P with black (resp. white) color contains one maximum (resp. minimum) of f_P and, in its small neighborhood with coordinates (x_1, x_2), f_P is locally given by $f_P(x_1, x_2) = x_1^2 + x_2^2$ (resp. $f_P(x_1, x_2) = -x_1^2 - x_2^2$);
(3) each double point of P corresponds to a saddle of f_P and, in its small neighborhood with coordinates (x_1, x_2), f_P is locally given by $f_P(x_1, x_2) = x_1^2 - x_2^2$;
(4) there are no singularities of f_P in D_n other than those in (2) and (3);
(5) if the outside boundary of an exterior region of P is exactly a component of ∂D_n then the outside boundary is contained in a level set of f_P;
(6) if an exterior region of P is not in case (5) then there is just one point, in its outside boundary, at which a level set of f_P intersects ∂D_n tangentially.

For each admissible divide we can describe level sets as shown in Figure 1. The level sets determine a Morse function f_P of P satisfying the above conditions.

For defining a complex valued function F_P we prepare a few notations. Let a_i, $i = 1, \cdots \ell$, be the critical points of f_P. Suppose that $U_i = \{x \in D_n \mid |x - a_i| < r_i\}$, for some $r_i > 0$, is the small neighborhood of a_i introduced in the condition (2) or (3) above, and set $U'_i = \{x \in D_n \mid |x - a_i| < r'_i\}$, where $r_i > r'_i > 0$. Let $\chi : D_n \to [0, 1]$ be a positive C^∞-differentiable bump function such that $\chi(x) = 0$ for $x \in D_n \setminus \cup_{i=1}^{\ell} U_i$ and $\chi(x) = 1$ for $x \in \cup_{i=1}^{\ell} U'_i$.

*The condition 'the outside boundary in that region is connected', which was not mentioned in [I], is necessary for drawing the level sets of an associated Morse function f_P.

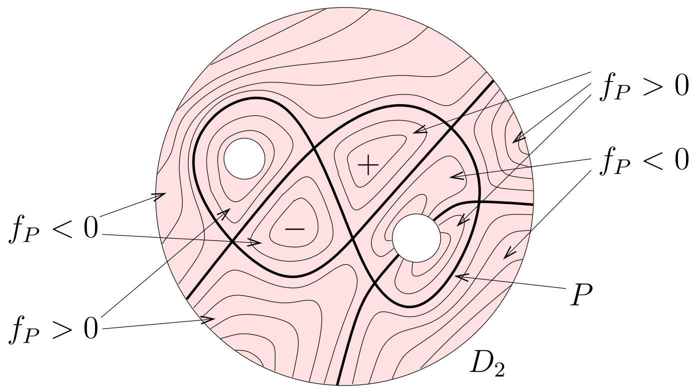

FIGURE 1. An example of an admissible divide P in D_2 with level sets which determine a Morse function f_P associated with P.

The complex valued function F_P is a map from $T(\mathbb{R}^2)$, which is the tangent bundle to $\mathbb{R}^2$, to $\mathbb{C}$ defined by

$$F_P(x,u) := f_P(x) + i df_P(x)(u) - \frac{1}{2}\chi(x)H_{f_P}(x)(u,u), \tag{3.1}$$

where $x \in \mathbb{R}^2$, $T_x(\mathbb{R}^2)$ is the set of tangent vectors to $\mathbb{R}^2$ at x, $u \in T_x(\mathbb{R}^2)$, $i = \sqrt{-1}$, $df_P(x)$ is the differential and $H_{f_P}(x)$ is the Hessian of f_P at $x \in \mathbb{R}^2$. By setting (x, u) to be the complex valued coordinates $x + iu$ in $\mathbb{C}^2$ as before, we can regard $T(\mathbb{R}^2)$ as $\mathbb{C}^2$ and F_P as a function from $\mathbb{C}^2$ to $\mathbb{C}$.

Recall that M_n is the 3-manifold $\#nS^1 \times S^2$ embedded in $\mathbb{C}^2$.

Definition 3.4. The *link* $L(P)$ of an admissible divide P in D_n is the set in M_n defined by

$$L(P) := M_n \cap F_P^{-1}(0). \tag{3.2}$$

It can also be defined by

$$L(P) := \{x + iu \in M_n \mid x \in P,\, u \in T_x(P)\}, \tag{3.3}$$

where $T_x(P)$ is the set of tangent vectors to P at $x \in D_n$.

The coincidence of (3.2) and (3.3) is proved in [I, Lemma 4.1]. The expression (3.2) is important for observing the Lefschetz fibration $F_P : N_\delta(D_n) \to \mathbb{C}$, while the expression (3.3) will be used when we need to see the position of the link $L(P)$ relative to the contact structure ξ.

Theorem 3.5 ([AC3, I]). *There is a 3-manifold $\hat{M}_n$, obtained by an embedded isotopy of M_n in $\mathbb{C}^2$ fixing the set $L(P)$, such that the argument map $\pi : \hat{M}_n \setminus L(P) \to S^1$ defined by $\pi := F_P/|F_P|$ is a locally trivial fibration over S^1 and its monodromy diffeomorphism is a product of positive Dehn twists. In particular, it gives a positive open book decomposition of M_n with $L(P)$ the binding.*

To prove our main theorem, we need an explicit description of a fiber surface of the open book decomposition in Theorem 3.5. In the rest of this section we describe it according to the observation in [AC4].

Let P be an admissible divide in D_n. The fiber surface over $1 \in S^1$ is determined by the equation $F_P/|F_P| = 1$, which is equivalent to the condition that $F_P(x+iu)$ is real and positive. We denote the fiber surface by F_1. Put

$$P_+ := \{x \in D_n \setminus \partial D_n \mid f_P(x) > 0,\, df_P(x) \neq 0\}.$$

The level sets of f_P define oriented foliations F_+ and F_- on P_+ such that $x + iu \in F_+$ (resp. $\in F_-$) if $df_P(x)(iu) > 0$ (resp. < 0). Put

$$P_{+,+} := \{x + iu \in M_n \mid x \in P_+,\, u \in T(F_+)\}$$

and

$$P_{+,-} := \{x + iu \in M_n \mid x \in P_+,\, u \in T(F_-)\},$$

where $T(F_\pm)$ is the set of tangent vectors on P_+ lying in the same direction as $F_\pm$. Put

$$F_M := \{x + iu \in M_n \mid x = M\}$$

for each maximum M in P_+, and also put

$$F_{s,+} := \{x + iu \in M_n \mid x = s,\, H_{f_P}(x)(u,u) < 0\}$$

for each double point s of P, which is a saddle point of f_P. Finally put

$$\partial D_+ := \{x \in \partial D_n \mid f_P(x) > 0\}.$$

Then the fiber surface F_1 is given by the union

$$F_1 = P_{+,+} \cup P_{+,-} \cup \partial D_+ \cup \bigcup_{s\in P} F_{s,+} \cup \bigcup_{M \in P_+} F_M,$$

where the gluings are ruled as shown in Figure 2. In the figure, R is an interior region while S may be an exterior region, and $R_\pm$ (resp. $S_\pm$) is the lift of R (resp. S) corresponding to the foliation $F_\pm$.

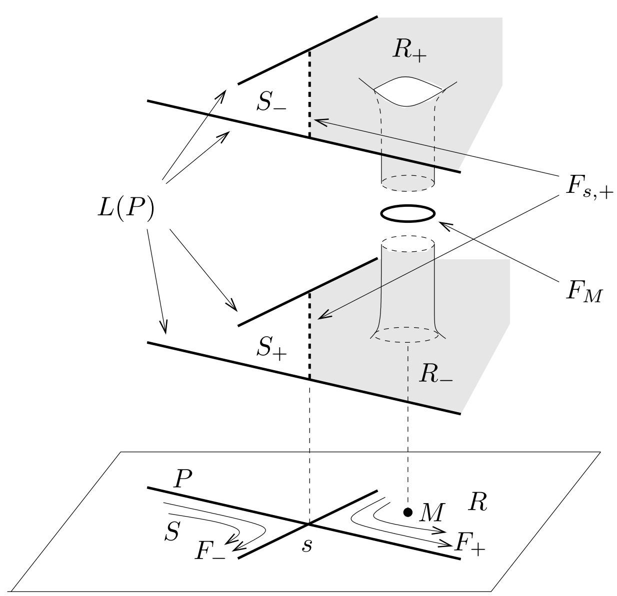

FIGURE 2. Building up the fiber F_1.

4. Oriented divides

Before starting a proof of the main theorem, we introduce oriented divides in D_n and their links defined in $\#nS^1 \times S^2$. These will be used for representing the given link L in the main theorem as we did in [Gi-I] and [I].

Definition 4.1. An *oriented divide* $\vec{P}$ in D_n is the image of a generic immersion of a finite number of copies of the oriented unit circle into D_n. Here the generic conditions are (i) and (iii) in Definition 3.1.

Definition 4.2. The *link* $L(\vec{P})$ of an oriented divide $\vec{P}$ in D_n is the set in M_n defined by

$$L(\vec{P}) := \{x + iu \in M_n \mid x \in P,\, u \in T_x(\vec{P})\},$$

where $T_x(\vec{P})$ is the set of tangent vectors to the immersed curve of $\vec{P}$ at x whose direction is consistent with the orientation of $\vec{P}$ at x.

5. Proof of Theorem 1.1

The key point of the proof of main theorem is to apply the technique of Akbulut and Ozbagci in [Ak-O] to the fiber surfaces of positive open book decompositions of $\#nS^1 \times S^2$ associated with divides in D_n.

We first state a theorem of Eliashberg which characterizes compact Stein surfaces in words of 2-handle attachings. Let W_n be a 4-manifold obtained from 4-ball by attaching n copies of 1-handles. Note that the boundary ∂W_n of W_n is homeomorphic to $\#nS^1 \times S^2$. Let K be a Legendrian knot in ∂W_n with respect to a contact structure. The *canonical framing* of K is the linking of the knot K and a knot K' obtained by pushing-off K into the direction normal to the 2-plane field of the contact structure.

The following is implicitly given by Eliashberg in [E]. The statement can be found in [Gom].

Theorem 5.1 ([E], cf. [Gom]). *An oriented, compact, smooth 4-manifold with boundary is a Stein surface if and only if it can be obtained from W_n, for some $n \geq 0$, by attaching 2-handles along Legendrian knots K_i, $i = 1, \cdots, m$, with respect to the standard contact structure in $\#nS^1 \times S^2$, with coefficient the canonical framing minus 1.*

Note that every Stein fillable 3-manifold can be obtained as the boundary of a compact Stein surface.

We need to introduce Lefschetz fibrations for relating the above handle attachings to positive open book decompositions. Let W be a compact, connected, oriented, smooth 4-manifold. A *Lefschetz fibration* $p : W \to \Delta$ over a disk Δ is a map such that

- each critical point a of p lies in the interior of W and admits a coordinate neighborhood with complex valued coordinates $z := (z_1, z_2)$ consistent with the given orientation of W;
- in a small neighborhood of each critical point a, the map p is locally given by the form $p(z) = p(a) + z_1^2 + z_2^2$;
- the orientation of $p(z)$ is consistent with that of Δ;

- each fiber $p^{-1}(c)$, $c \in \Delta \setminus \partial\Delta$, intersects ∂W transversely;
- $p^{-1}(\partial\Delta)$ is contained in ∂W.

A Lefschetz fibration is called *allowable* if all its vanishing cycles are homologically non-trivial in the fiber. Note that a Lefschetz fibration $p : W \to \Delta$ induces, on its boundary ∂W, a positive open book decomposition of ∂W with binding the set $p^{-1}(0) \cap \partial W$.

The total space of a Lefschetz fibration has a canonical handle decomposition characterized by the critical values of the fibration map, see [K, Ak-O]. The following lemma gives a method for making Lefschetz fibrations by using 2-handle attachings.

Lemma 5.2 ([Ak-O] Remark 1)**.** *Let $p : W \to \Delta$ be an allowable Lefschetz fibration over a disk Δ. Suppose that W' is obtained from W by attaching a 2-handle with coefficient the product framing minus 1. Then this 2-handle attaching induces an allowable Lefschetz fibration $p' : W' \to \Delta'$ over a disk Δ'.*

In [Ak-O], Akbulut and Ozbagci gave a proof of the existence of a positive open book decomposition for every Stein fillable 3-manifold. The main idea in their proof is to find a good position of the fiber surface of a torus knot in S^3 such that, for each $i = 1, \cdots, m$, K_i stays on the surface and the product framing of K_i coincides with $tb(K_i)$. We will use the same strategy later.

Now we give a proof of the main theorem.

Proof of Theorem 1.1. Let W be a Stein surface whose boundary is the prescribed Stein fillable 3-manifold, denoted by M. Let L denote the given link in M. We suppose that W is obtained from W_n by attaching 2-handles H_i along Legendrian knots K_i, with respect to the standard contact structure in ∂W_n, with coefficients $tb(K_i)-1$, where $i = 1, \cdots, m$. Using regular isotopy, we can isotope the link L in ∂W so that it does not intersect the 2-handles $\cup_{i=1}^m H_i$. Thus we can assume that L is a link in $(\#nS^1 \times S^2) \setminus (\cup_{i=1}^m K_i)$.

Recall that the 3-manifold M_n is the boundary of a small compact neighborhood of D_n in $\mathbb{C}^2$ and homeomorphic to $\#nS^1 \times S^2$. We have seen that M_n has the decomposition $M_n = \partial N_A \cup \partial N_B$ and the piece ∂N_B can be regarded as the unit tangent bundle to B, where $B \subset D_n$ is the complement of a small compact neighborhood of ∂D_n in D_n. As mentioned in Lemma 2.1, each Legendrian knot K_i has a wave front representative w_i on B. We set $K := \cup_{i=1}^m K_i$ and $w := \cup_{i=1}^m w_i$. We can also get an oriented divide representative of the link L on B.

Claim 5.3. *Every link in M_n has an oriented divide representative in B up to regular isotopy in $M_n \setminus K$.*

Proof. The claim is similar to Theorem 1.1 in [Gi-I] and can be proved by using the same technique (cf. [I, Ch-Gor-Mu]). We here explain roughly how to prove this claim. Let L denote the given link in M_n. Using regular isotopy, we assume that L is included in ∂N_B. If we think of ∂N_B as the unit tangent bundle to B, the link L corresponds to a continuous assignment of vectors with length $\delta > 0$ and based on a curve C in B. By setting L in a generic position, we can assume that the curve C consists of generically immersed circles in B. Assign an orientation to each component of C and isotope the vectors so that they are in the same direction as the orientation of C except over a finite number of short intervals. Here we first isotope the vectors in a neighborhood of the double points of C

and in a neighborhood of the intersection points of C and w and then isotope the vectors on the other parts so that the short intervals are included in $C \setminus (\{\textit{double points}\} \cup w)$. The proof is done by replacing each short interval by a small loop such that the move of tangent vectors to the small loop is consistent with the move of the vectors of the isotoped continuous assignment of vectors corresponding to the link L. $\square$

Thus we have a wave front representative w of K and an oriented divide representative, denoted by $\vec{P}$, of L on B. Using Legendrian isotopy for K and regular isotopy for L, we can assume that the union of immersed curves of these representatives has only node and cusp singularities. Note that the co-orientation of w and the orientation of $\vec{P}$ can not be in the same direction at each intersection of the immersed curves of w and $\vec{P}$ since the links K and $L = L(\vec{P})$ do not intersect each other.

Now we regard the immersed curve of $\vec{P}$ as a divide in D_n and denote it by P. We then add another divide P' to P so that $P \cup P'$ satisfies the admissible conditions. Let $a_1, \cdots, a_r$ be the intersection points of w and $\vec{P}$, $b_1, \cdots, b_s$ the self intersection points of w, and let $c_1, \cdots, c_t$ be the cusp singularities of w. Let U_j (resp. V_k and W_ℓ) be a sufficiently small neighborhood of a_j (resp. b_k and c_ℓ).

Next we make a new divide $\hat{P}$ from $P \cup P'$ according to the following rules:

(1) add an immersed curve parallel to $P \cup P'$;
(2) outside W_ℓ, write two immersed curves parallel to w in such a way that w lies between the two parallel curves;
(3) in U_j, if the argument between the orientation of $\vec{P}$ and the co-orientation of w is less than $\pi/2$ and $\vec{P}$ stays in the left-side line with respect to the co-orientation of w, then add two crossings to the vertical two parallel curves near U_j, as shown on the left and center in Figure 3, so that $\vec{P}$ stays in the right-side line in U_j;
(4) if a region of the divide obtained according to steps (1) and (2) intersects a neighborhood U_j and the boundary ∂D_n then add a crossing to the two parallel curves as shown on the right in Figure 3;
(5) if a region of the divide obtained intersects two neighborhoods V_k and $V_{k'}$ (possibly $V_k = V_{k'}$) then add a crossing to the two parallel curves as shown on the left in Figure 4;
(6) inside each W_ℓ, we define $\hat{P}$ as shown on the right in Figure 4.

Note that the divide $\hat{P}$ is admissible and contains $P \cup P'$. In particular, the link $L = L(\vec{P})$ is still contained in the binding of the positive open book decomposition associated with the divide $\hat{P}$.

We remark that $\hat{P}$ and w may intersect orthogonally in V_k. In this case, since the co-orientation of w is tangent to $\hat{P}$ at the orthogonal intersection point, the link K intersects a component of $L(\hat{P})$, but this is not a component of $L = L(\vec{P})$ since $\vec{P}$ does not pass through the neighborhood V_k.

By changing the sign of $f_{\hat{P}}$ if necessary, we assume that $f_{\hat{P}} > 0$ in the regions between the two parallel curves described in the above steps outside U_j, V_k and W_ℓ. Let F_1 denote the fiber surface, over $1 \in S^1$, of the positive open book decomposition associated with the admissible divide $\hat{P}$. As explained in Section 3, the surface F_1 can be understood as the closure of the foliations $F_\pm$ on the regions P_+.

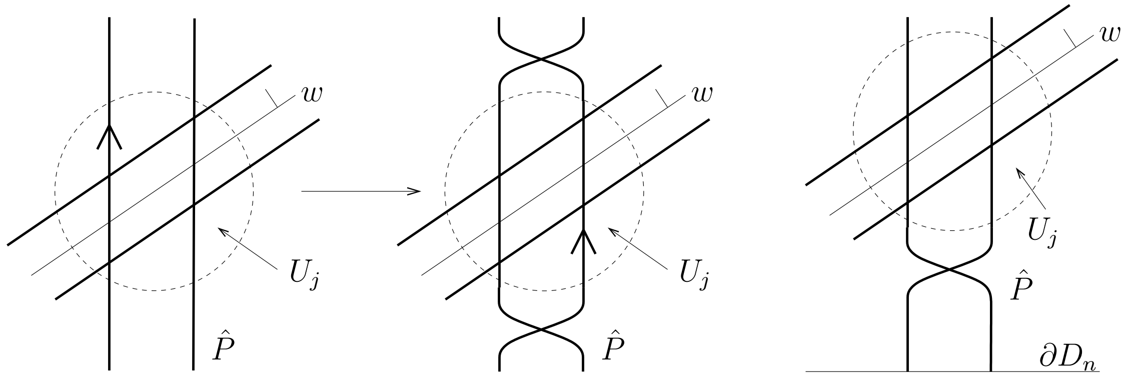

FIGURE 3. The left and center figures show the addition of two crossings in both sides of U_j. The divide $\hat{P}$ is represented by thick curves and w is represented by co-oriented thin ones. The orientation written on the curve of $\hat{P}$ represents that of $\vec{P}$. The right figure shows the addition of a crossing between U_j and ∂D_n.

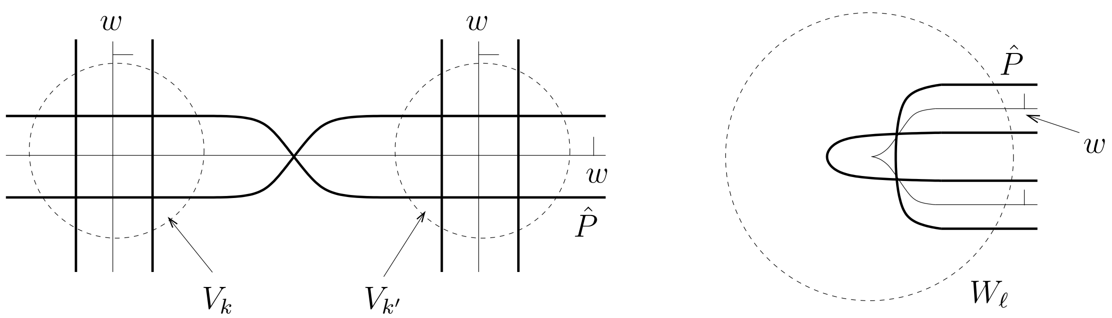

FIGURE 4. The left figure shows the addition of a crossing to the two parallel curves between V_k and $V_{k'}$. The right figure is the divide $\hat{P}$ inside W_ℓ.

We will move the surface F_1 so that the link K stays on it. During the move, $L = L(\vec{P})$ can not intersect K though the other components of $L(\hat{P})$ can intersect it. Since it is difficult to draw figures of the surface moved, we describe the position of K relative to the moved surface, that is, we fix the divide $\hat{P}$ and its foliations $F_\pm$, which represent the surface F_1, and describe the move of K by a family $\hat{w}_t$, $t \in [0,1]$, of immersed curves with continuous assignment of vectors such that $\hat{w}_0 = w$, where the continuous assignment of vectors is the co-orientation of w when $t = 0$ but it is not necessary to be orthogonal to the immersed curves when $t > 0$. We call such a curve a *swinging wave front representative*.

Since $L = L(\vec{P})$ can not intersect K during the move, the swinging co-orientation of $\hat{w}_t$ can not lie in the same direction as the orientation of $\vec{P}$ at each intersection point of $\vec{P}$ and $\hat{w}_t$. Also, at each self intersection point of $\hat{w}_t$ the two swinging co-orientations can not be in the same direction otherwise the isotopy type of K may change.

Now we move the surface F_1 to another surface $\hat{F}_1$ under the above rules so that K is realized, at $t = 1$, by the swinging wave front representative $\hat{w}$ $(= \hat{w}_1)$ described in the following: If there exists the curve of w in an interior region not included in U_j, V_k and

W_ℓ then we set $\hat{w}$ as shown in Figure 5. Here we do not need to swing the co-orientation, i.e. we do not need to move the surface F_1. The dots represent the maxima of the interior regions and the dotted circles centered at the maxima are the level sets of $f_{\hat{P}}$. The wave front representative $\hat{w}$ is, at each point, either tangent to a level set of $f_{\hat{P}}$ or passing though a maximum. Hence the link K is on the fiber surface $\hat{F}_1$ in this part.

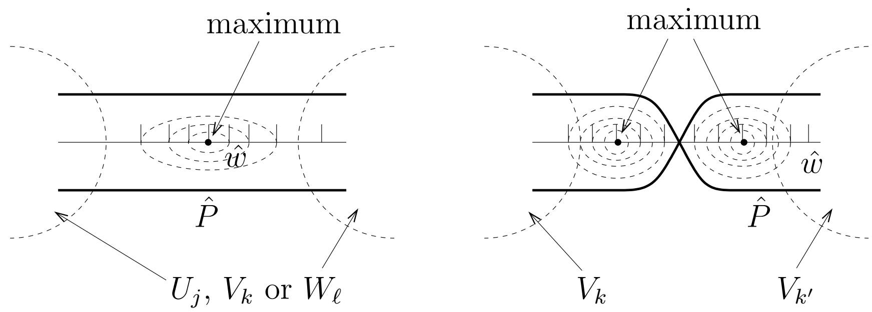

FIGURE 5. The swinging wave front representative $\hat{w}$ outside U_j, V_k and W_ℓ.

We set $\hat{w}$ inside U_j as shown on the left in Figure 6. If w runs from top-left to bottom-right in U_j then we need to consider the mirror image of the figure. For making the figure to be simple, we did not draw the level sets of $f_{\hat{P}}$, cf. Figure 5. We set the swinging co-orientations of $\hat{w}$ to be tangent to the level sets of $f_{\hat{P}}$ outside the quadratic singularities. Then by observing $\hat{w}$ and $F_\pm$ we can conclude that K lies on $\hat{F}_1$ in this part.

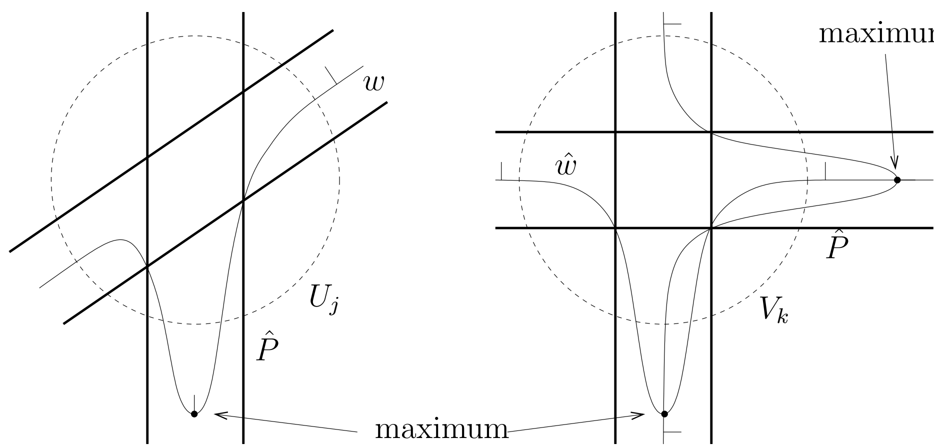

FIGURE 6. The swinging wave front representative $\hat{w}$ inside U_j and V_k.

We need to check that there is no intersection of K and L during the move $\hat{w}_t$. An intersection of K with $L(\hat{P})$ occurs if the swinging co-orientation of w_t and the tangent direction to $\hat{P}$ coincide at an intersection point of $\hat{w}_t$ and $\hat{P}$. By observing the family $\hat{w}_t$ in U_j, we can verify that the intersection occurs once over the left vertical curve of $\hat{P}$ described on the left in Figure 6. Since we have set in step (3) that the curve of $\vec{P}$ is the right vertical line in the figure, we can conclude that there is no intersection of K and $L = L(\vec{P})$ over U_j during this move.

We also need to check if there really exists the maximum which $\hat{w}$ passes and if there is no conflict with other neighborhoods. First note that the region containing the maximum does not connect to V_k and W_ℓ since $\hat{w}$ does not passes though U_j vertically in the figure. If the region is connected to the boundary ∂D_n then there is a maximum due to step (4). The rest is the case where it is connected to a neighborhood $U_{j'}$. If $U_j = U_{j'}$ then the union of the region and U_j constitutes an embedded annulus in D_n and $\hat{w}$ intersects it once transversely, which contradicts the fact that $\hat{w}$ consists of closed curves. Hence we can assume that $U_j \neq U_{j'}$. If the co-orientation of w in $U_{j'}$ is in the same direction as that of w in U_j (along the vertical curves of $\hat{P}$ in the figure) then $\hat{w}$ in $U_{j'}$ passes through a maximum different from the one corresponding to U_j. Hence we can observe U_j and $U_{j'}$ independently. If the co-orientation of w in $U_{j'}$ is in the opposite direction to that of w in U_j then $\hat{w}$ in $U_{j'}$ passes through the same maximum as the one corresponding to U_j. But we can also observe U_j and $U_{j'}$ independently since the co-orientations are in the opposite directions.

Inside V_k, we set $\hat{w}$ as shown on the right in Figure 6. The figure of V_k is in the case where the co-orientation of the vertical component of w is in the right direction. If it is opposite then we need to consider the mirror image of the figure. We can see that K lies on $\hat{F}_1$ as in the case of U_j and that there is no intersection of K and L during the move $\hat{w}_t$ since $\vec{P}$ does not pass through a neighborhood of type V_k. We can also check that the two strands of K corresponding to the vertical and horizontal curves of $\hat{w}$ in V_k do not intersect each other during this move. Since w passes between both the vertical and horizontal parallel curves of $\hat{P}$ in V_k, these curves connect to either the horizontal curves in U_j on the left in Figure 6, or parallel curves in $V_{k'}$ (possibly $V_k = V_{k'}$), or those in W_ℓ. Obviously, the maxima which $\hat{w}$ passes do not conflict with those corresponding to U_j, and they also do not conflict with $V_{k'}$ by the setting in step (5). Hence we can observe each V_k independently.

Inside W_ℓ, we set $\hat{w}$ as shown in Figure 7. The figure is in the case where the co-orientation is in the upper direction. If it is opposite then we change the co-orientation in the figure into the opposite direction. As in the previous cases, we can easily check that K lies on $\hat{F}_1$ in the figure.

Thus we got the move from F_1 to $\hat{F}_1$ without intersection of K and $L = L(\vec{P})$ such that K lies on the surface $\hat{F}_1$.

The product framing of K_i associated with the Lefschetz fibration is the framing of K_i obtained by pushing-off it in the direction normal to the fiber surface $\hat{F}_1$. We now prove that this framing coincides with the canonical framing of K_i.

Claim 5.4. *For each $i = 1, \cdots, m$, the product framing of K_i coincides with the canonical framing of K_i.*

Proof. Let v be a component of $\hat{w}$. We assign an orientation to v and put the labels $p_1, q_1, p_2, q_2, \cdots, p_r, q_r$ to the maxima of $f_{\hat{P}}$ and the cusps and double points of $\hat{P}$ according to the orientation. Since the double points appear at every other label, we can assume that p_i is either a maximum or a cusp, and q_i is a double point. We then separate v into r intervals $I_1, \cdots, I_r$ in such a way that I_i contains the pair of points p_i and q_i.

We first prove the coincidence of the framings for each interval I_i with p_i a maximum. Instead of pushing-off the link of v in the direction normal to $\hat{F}_1$, we shift it on $\hat{F}_1$ into

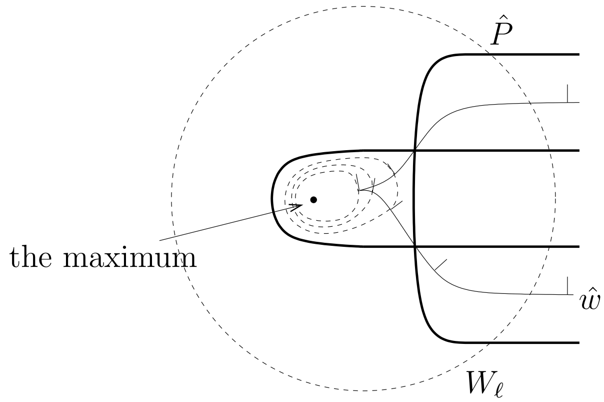

FIGURE 7. The swinging wave front representative $\hat{w}$ inside W_ℓ.

one of the possible directions. This can be represented by using a swinging wave front representative as shown on the top in Figure 8. This swinging wave front representative can isotope as shown on the bottom in the figure. By setting the isotoped swinging co-orientation to be orthogonal to the immersed curve, we can see that the curve is the one obtained by pushing-off the swinging wave front representative $\hat{w}$ in the direction of the co-orientation, which means that the corresponding link can be obtained from the link of $\hat{w}$ by pushing-off it in the direction normal to the 2-plane field ξ. Hence the two framings coincide in this part.

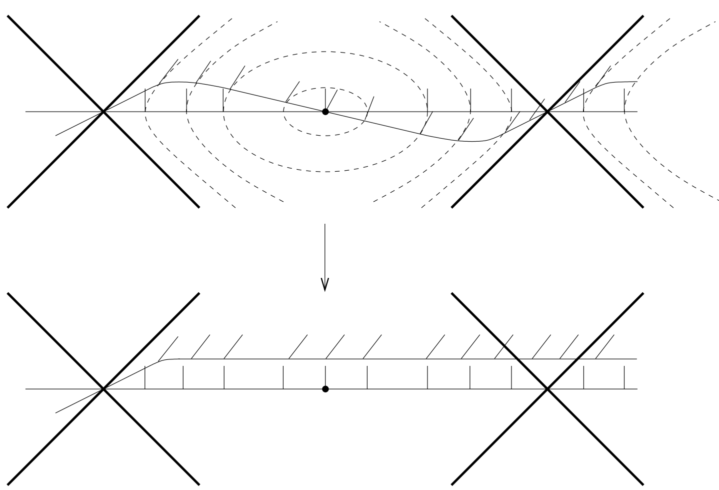

FIGURE 8. Shift $\hat{w}$ on $\hat{F}_1$.

Around each V_k there are three self intersection points of $\hat{w}$ as seen on the right in Figure 6, and there is no other self intersection of $\hat{w}$ in D_n. For each V_k, we can consider the two curves of $\hat{w}$ independently since the difference of the co-orientations of these two curves are at least $\pi/2$ degree and we need to swing them a very little during the shift in Figure 8. Hence the coincidence of the framings follows from the above observation.

The coincidence of the framings for each interval I_i with p_i a cusp can be proved by the same argument. The shift of v on $\hat{F}_1$ is represented as shown in Figure 9 (a). The

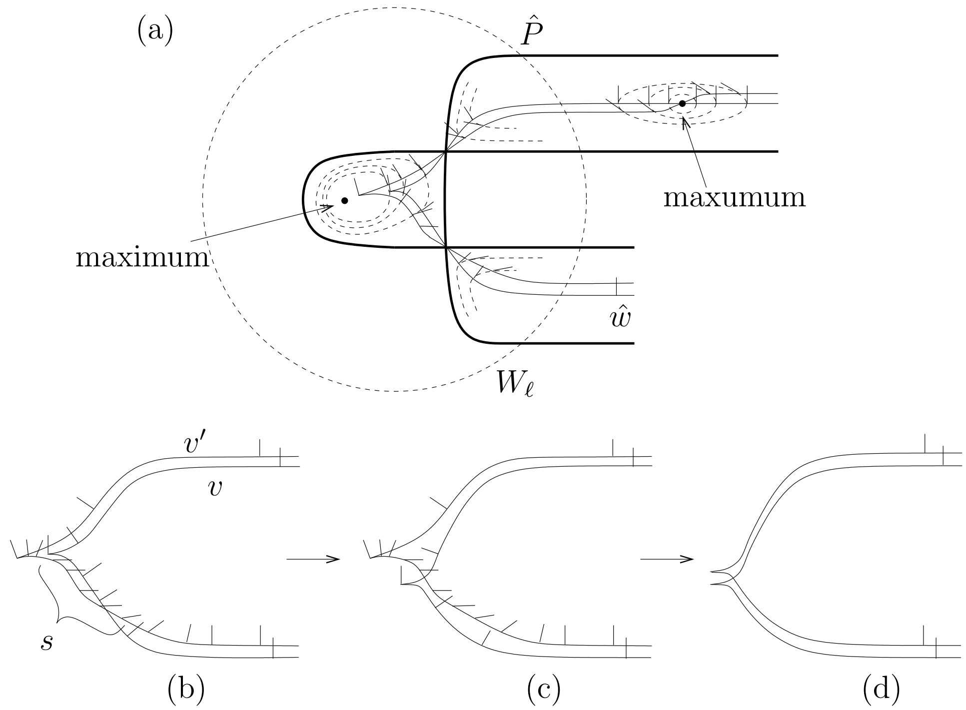

FIGURE 9. Shift $\hat{w}$ on $\hat{F}_1$ near a cusp.

isotopy move of the swinging wave front representative from (a) to (b) is the same move as in Figure 8. Denote by v' the swinging wave front representative obtained by the shift, see the figure (b). We assume that on the segment s in v' specified in the figure (b) the swinging co-orientation lies in the right direction. Then we move the curve of v so that the cusp comes under the cusp of v'. This move does not change the linking of the links of v and v' since all the swinging co-orientations on s lie in the right direction and no swinging co-orientation of v lies in that direction. Thus we obtain the figure (c). Finally we obtain the figure (d) by swinging the co-orientations of v and v' so that they are orthogonal to the immersed curves of v and v' respectively. This move does not change the linking of the links of v and v' since the two co-orientations at the intersection point of v and v' do not intersect each other during the move. Thus we can see that the curve is the one obtained by pushing-off the swinging wave front representative $\hat{w}$ in the direction of the co-orientation, and hence the two framings coincide. □

We continue the proof of main theorem. In [I] we introduced the Lefschetz fibration associated with a divide in D_n and obtained the fibration in Theorem 3.5 on its boundary, and we can easily check that this Lefschetz fibration is allowable. Hence, by the coincidence of the two framings in Claim 5.4, we can conclude that the 2-handle attachings along w, which yield the compact Stein surface W, also produce a new allowable Lefschetz fibration by Lemma 5.2. The boundary of W is the Stein fillable 3-manifold M expected and it has a positive open book decomposition associated with the Lefschetz fibration. As already mentioned, the link L is contained in the binding of the positive open book decomposition of M_n associated with $\hat{P}$. Since the 2-handle attachings do not change the binding, L is still contained in the binding of the positive open book decomposition of M

associated with the Lefschetz fibration. Suppose $L \cup L'$ is the binding. Then, as we did in [I], we can modify the positive open book decomposition by plumbing positive Hopf bands so that the binding is the union of L and a knot in $M \setminus L$ (cf. [S, Ha, Ga]). This completes the proof of Theorem 1.1. □

6. A conjecture

The main theorem of this paper suggests that we can freely choose the bindings of positive open book decompositions of Stein fillable 3-manifolds and the level of freedom is higher than the complexity of links in the 3-manifolds. Since every closed, oriented, smooth 3-manifold has an open book decomposition[Al], it is natural to ask if any 3-manifold has such freedom of the bindings. We here propose a conjecture concerning this question.

Conjecture 6.1. *Let $\mathcal{M}_k$, $k \geq 0$, be the set of closed, oriented, smooth 3-manifolds which have open book decompositions whose monodromies contain at most k negative Dehn twists. Suppose that a 3-manifold M is in $\mathcal{M}_k \setminus \mathcal{M}_{k-1}$ (here we set $\mathcal{M}_{-1} = \emptyset$). Then for any link L in M there exists a knot L' such that $L \cup L'$ is the binding of an open book decomposition of M whose monodromy contains exactly k negative Dehn twists.*

When $k = 0$, $\mathcal{M}_0$ is the set of Stein fillable 3-manifolds and hence the conjecture is true by our main theorem. However we know nothing about the class $\mathcal{M}_k \setminus \mathcal{M}_{k-1}$ for $k \geq 1$. For instance, it is already an interesting problem to ask the minimal numbers of negative Dehn twists of open book decompositions of the non-Stein fillable 3-manifolds known in [Lis1, Lis2].

References

[AC1] N. A'Campo, *Le groupe de monodromie du déploiement des singularités isolées de courbes planes I*, Math. Ann. **213** (1975), 1-32.

[AC2] N. A'Campo, *Le groupe de monodromie du déploiement des singularités isolées de courbes planes II*, Actes du Congrès International des Mathematiciens, Vancouver, 1974, pp. 395-404.

[AC3] N. A'Campo, *Real deformations and complex topology of plane curve singularities*, Annales de la Faculté des Sciences de Toulouse **8** (1999), no. 1, 5-23.

[AC4] N. A'Campo, *Generic immersions of curves, knots, monodromy and gordian number*, Publ. Math. de l'I.H.E.S. **88** (1998), 151-169.

[Ak-O] S. Akbulut, B. Ozbagci, *Lefschetz fibrations on compact Stein surfaces*, Geometry & Topology **5** (2001), 319-334.

[Al] J. Alexander, *A lemma on systems of knotted curves*, Proc. Nat. Acad. Sci. U.S.A. **9** (1923), 93-95.

[B] D. Bennequin, *Entrelacements et équations de Pfaff*, Astérisque **107-108** (1983), 87-161.

[Ch-Gor-Mu] C. Chmutov, V. Goryunov, H. Murakami, *Regular Legendrian knots and the HOMFLY polynomial of immersed plane curves*, Math. Ann. **317** (2000), no. 3, 389-413.

[E] Y. Eliashberg, *Topological characterization of Stein manifolds of dimension* > 2, Internat. J. Math. **1** (1990), 29-46.

[E2] Y. Eliashberg, *Contact 3-manifolds twenty years since J. Martinet's work*, Ann. Inst. Fourier **42** (1992), 165-192.

[Ga] D. Gabai, *The Murasugi sum is a natural geometric operation*, Low-dimensional topology (San Francisco, Calif., 1981), Contemp. Math. 20, A.M.S. Providence RI, 1983, pp. 131-143.

[Gi-I] W. Gibson, M. Ishikawa, *Links of oriented divides and fibrations in link exteriors*, Osaka J. Math. **39** (2002), 681-703.

[Gom] R.E. Gompf, *Handlebody construction of Stein surfaces*, Ann. of Math. **148** (1998), 619-693.

[GZ1] S.M. Gusein-Zade, *Intersection matrices for certain singularities of functions of two variables*, Funct. Anal. Appl. **8** (1974), 10-13.

[GZ2] S.M. Gusein-Zade, *Dynkin diagrams of singularities of functions of two variables*, Funct. Anal. Appl. **8** (1974), 295-300.

[GZ3] S.M. Gusein-Zade, *The monodromy groups of isolated singularities of hypersurfaces*, Russian Math. Surveys **32** (1977), 23-69.

[Ha] J. Harer, *How to construct all fibered knots and links*, Topology **21** (1982), 263-280.

[I] M. Ishikawa, *Tangent circle bundles admit positive open book decompositions along arbitrary links*, to appear in Topology.

[K] A. Kas, *On the handlebody decomposition associated to a Lefschetz fibration*, Pacific J. Math. **89** (1980), 89-104.

[Lis1] P. Lisca, *Symplectic fillings and positive scalar curvature*, Geometry & Topology **2** (1998), 103-116.

[Lis2] P. Lisca, *On symplectic fillings of* 3*-manifolds*, Proceedings of 6th Gökova Geometry-Topology Conference. Turkish J. Math. **23** (1999), 151-159.

[Lo-Pi] A. Loi, R. Piergallini, *Compact Stein surfaces with boundary as branched covers of* B^4, Invent. Math. **143** (2001), 325-348.

[S] J.R. Stallings, *Constructions of fibred knots and links*, A.M.S. Proc. Symp. in Pure Math. 32, 1978, pp. 55-60.